Natural Science

T. Whitfield

Modelling an Initial Public Offering Using Action Theory

Bachelor Thesis

GRIN

Bibliographic information published by the German National Library:

The German National Library lists this publication in the National Bibliography; detailed bibliographic data are available on the Internet at http://dnb.dnb.de .

Imprint:

ISBN: 9783346031525

This book at GRIN:

https://www.grin.com/document/500329

T. Whitfield

Modelling an Initial Public Offering Using Action Theory

GRIN Verlag

Modelling an Initial Public Offering using Auction Theory

Author: T. Whitfield

September 5, 2019

Abstract

This paper examines the behaviour of bidders in an initial public offering auction (IPO) when a seller is offering a perfectly divisible share to potential buyers. Due to the massive effect and news coverage IPOs have on the modern day economy, this paper aims to unearth some of the mystery behind such auctions. It focuses mostly on the behaviour of bidders in a complete and incomplete information setting, where a bidder's incentives change given their information and competition. Using the theoretical framework of the paper, I then apply the findings to the real world and comment on the realism of the models. Lastly, I recommend possible extensions to the models in order to bring a new dimension to the analysis.

Contents

List of Figures

List of Tables

Introduction

An initial public offering is a major stepping stone for any company as it evolves from a private to a publicly traded company. The process of going public is a complicated procedure, so they often solicit the help from an investment bank to help price the stock as well as find suitable investment. The purpose of an IPO is to help the selling company gather capital in order to expand operations. They do this by offering equity to potential investors in the form of a share which can be traded publicly via the necessary stock market. There are currently sixteen stock exchanges around world with a market capitalization of over $1 trillion each, the biggest being the New York Stock Exchange which has a market capitalization of over $28.5 trillion which is larger than the US gross domestic product for 2018 [9] [12].

This paper hopes to shine some light on the process of taking a company public by unearthing a bidder's behaviour and incentives under a range of different situations. Applying auction theory to an initial public offering is relatively unexplored in economic journals, so this paper aims to extend work already presented in the literature and give insightful explanations to problems which have presented themselves to sellers and buyers. These problems include but are not limited to the peculiar situation in which it is common that the stock price at the end of the first day of trading for a newly publicised company is higher than its offering price [8]. However, according to UBS, around 60% of IPOs provide investors with negative returns in their first five years, suggesting an extraordinary hype in first day trading compared to a more rational long term view. Some researchers and analysts claim this is due to the conflict banks face when pricing the company. This conflict arises from the bank's desire to raise as much money for the client as possible meanwhile offering a low price to encourage people to purchase the shares and increase the flow of shares where the bank can gather profit from trading expenses [11]. Therefore, it is the job of this paper to apply a theoretical framework to IPO auctions to find what conclusions can be made in a simplified environment.

One of the most important aspects of an initial public offering auction, is the fact it is a common value auction, where the company has a true value but bidders use the information available to them to estimate the value of the company. The information may come via financial reports such as the company's balance sheet, profit and loss accounts and cash flow statement alongside industry overview and various other economic factors. The models outlined in this paper use this fact to create the optimal bidding strategies for the bidders, whom do not want to fall victim to the winner's curse which is so prevalent in common value auctions. A famous example of the winner's curse is the Department of Transport's auction of the rights to operate the train network on the East coast of the United Kingdom. The winner of the auction has on three occasions gone bust from over-estimating the number of passengers over the lease.

The paper comprises of three main parts: the complete information model, incomplete information model, and the discussion and conclusion. In Part I as mentioned introduces a complete information model starting in the two person auction before generalizing to the n

bidder case. From this, we incorporate uncertainty into the model in Part II, where we have an auction with incomplete information. Part III then provides a brief overview of what could be introduced into the model to make it better resemble real world dynamics as well as summarising any findings found in the thesis.

Part I
Complete Information

1 Model I

In order to fully understand an Initial Public Offering auction, or share auction, we must focus on the complete information common value case, whereby each bidder knows with certainty the value of the company at auction. This allows us to understand the strategic decisions each bidder must consider on commencement of the auction. The first assumption we make in our model is that the seller places a value of zero on any unsold shares in the auction. We will assume this is due to external credit constraints which occur exogenous. The auction is between n potential buyers, denoted $\mathcal{B}_n = \{1, ..., \text{n}\}$, each with corresponding exogenous budget constraints, $\mathcal{W}_n = \{w_1, ..., w_n\}$. Without loss of generality, we assume $w_j \geq w_{j+1}, \forall j \in \mathcal{B}_n$ and $w_i \in [0, 1], \forall i \in \mathcal{B}_n$. Otherwise, the bidders are completely symmetric in terms of risk neutrality and access to information.

The auctioned item is one perfectly divisible share of a company which can only take one of two possible values. Let a 'Good' firm be worth £1 and a 'Bad' firm be worth nothing. Due to the seller's incentives and the buyer's information, the firm will always be sold, therefore the auction is socially optimal, as shown in section 1.4. Due to the fact we are dealing with a continuously divisible unit and not x discrete units, common bidding, payment and allocation rules are not applicable.

1.1 Bidding

The model I present requests the bidders to submit nonnegative and non-increasing demand schedules, $x_i(p)$, which specifies the percentage as a decimal of the firm each bidder would like for each price p. It is important to note at this stage that the demand schedules need not be continuous, which can be very restrictive, however, we do impose that each bidder submit a quantity demand of at most 1. Due to the framework, we can consider this to be an inverse demand function where each bidder reveals the quantity they demand for each price p. Furthermore, let

$$X(p) = \sum_{i \in \mathcal{B}_n} x_i(p)$$

denote the aggregate market demand for the firm at price p. In addition, in this auction, we assume if a bidder submits a quantity demanded and price bundle which violates their budget constraint, then that bidder will be executed and the auction run again - an effective deterrent against bidders promising the seller revenue they cannot afford.

1.2 Allocation

As the auction runs with a perfectly divisible share, the allocation rule must allow each winner in the auction to walk away with a percentage of the good. Another important aspect is that the sum of quantity allocated cannot be more than the total shares at auction, in this case, one. Therefore, we adopt the allocation rule:

$$y_i(p^*) = \frac{x_i(p^*)}{X(p^*)} = \frac{x_i(p^*)}{\sum_{i \in \mathcal{B}_n} x_i(p)}[5] \tag{1}$$

where by definition (2),

$$\sum_{i \in \mathcal{B}_n} y_i(p^*) = 1$$

This, in essence, extracts bidder i's quantity demanded and divides it by the aggregate demand at price p^*.

1.3 Payment

An important aspect of an auction is how the winning bidder(s) pay. The payment rule (2) used in this share auction is based on the pricing rule in Vijay Krishna's 'Auction Theory' for selling K identical objects in a uniform-price auction. The logic of using this pricing rule makes senses in an IPO setting as it means each bidder pays the same price p^* for each unit allocated, where p^* is calculated according to:

$$p^* = \max(p|X(p) \geq 1)[6] \tag{2}$$

Therefore, if bidder A wins $\alpha\%$ of the firm in the auction, then according to (2), they will pay αp^*.
A crucial facet to understand in this auction of complete information is that: whenever the firm being auctioned is bad, each bidder demands 100% of the firm for price $p = 0$, and zero thereafter. This fact can be generalised to n bidders, therefore, we do not need to consider this in our analysis due to the complete information element.

1.4 Social Planner's Problem

Before analysing the strategic decisions each bidder will take in the auction, we must analyse what outcome maximises social welfare. Let the social planner be indifferent between who receives the shares and for what price they are traded, but wants the firm to be sold. Let $E = \mathcal{B} \cup S = \{1, ..., n, S\}$ denote the set that contains the seller(S) and all potential bidders($\mathcal{B}_n$) in the auction and let each individual be risk neutral with utility function $U_i = s_i - p_i$, where s_i denotes the value of the shares i wins in the auction and p_i is how much bidder i paid for their s_i shares of the company. The decision which maximises social welfare is calculated as so:

$$
\begin{aligned}
\max_{\substack{s_i,p_i;\\ i\in\mathcal{B}_n}} SW &= \max_{\substack{s_i,p_i;\\ i\in\mathcal{B}_n}} \left[\sum_{i\in E} U_i\right],\ \text{subject to } s_1+...+s_n \leq 1,\ \text{where } s_1,...,s_n \in [0,1] \\
&= \max_{\substack{s_i,p_i;\\ i\in\mathcal{B}_n}} [U_1+...+U_n+U_S] \\
&= \max_{\substack{s_i,p_i;\\ i\in\mathcal{B}_n}} [s_1-p_1+...+s_n-p_n+p_1+...+p_n] \\
&= \max_{\substack{s_i,p_i;\\ i\in\mathcal{B}_n}} [s_1+...+s_n] \\
&= 1
\end{aligned} \tag{3}
$$

This tells us that the outcome which maximises social welfare is if all shares are sold in the auction so 100% of company is allocated among the n bidders in $\mathcal{B}_n$.

1.5 Two Bidder Model

When the firm is known with certainty to be Good, for simplicity, we begin with two potential buyers, labelled bidder 1 and bidder 2 with the same assumption of $w_1 \geq w_2$. With our core rules of the auction and players $\mathcal{B}_2 = \{1,2\}$ identified, the remaining fundamentals to be established are the strategies and payoffs for each player. Let π_1 and π_2 denote the profit functions of each bidder in the auction, which can be generally defined to be:

$$
\pi(p^*) = \underbrace{(1-p^*)}_{\text{Profit/unit}}\ \underbrace{y_i(p^*)}_{\text{Allocation}} \tag{4}
$$

$$
= (1-p^*)\frac{x_i(p^*)}{X(p^*)} \tag{5}
$$

where (4) uses both the pricing rule and allocation rule in its formulation.
By construction, as bidder 1 has a budget constraint of at least w_2, this bidder has an advantage as they act as a price setter in the auction as bidder 2 cannot demand the whole company for price $p > w_2$ without being executed. Thus, bidder 1 and bidder 2 can submit demand schedules:

$$
x_1(p) = \begin{cases} 1 & p \leq w_2 \\ 0 & p > w_2 \end{cases}
$$

$$
x_2(p) = \begin{cases} 1 & p \leq w_2 \\ \frac{w_2}{p} & w_2 < p \leq 1 \\ 0 & p > 1 \end{cases}
$$

where bidder 1 demands the maximum up to w_2 and zero thereafter, and bidder 2 also demands 100% of the firm up to w_2 and a quantity for each price which binds their budget constraint up to $p = 1$. This means bidder 1 and 2 can share 50% of the company each and pay $\frac{w_2}{2}$, in which case both making a profit of $0.5(1-\frac{w_2}{2})$. However, in this case, other than setting the price of the auction, bidder 1 does not exert their full strength as the wealthier buyer. Hence, we must analyse bidder 1's incentives when deciding the price for the company. To do this, we must introduce a new choice variable ϵ for bidder 1 which controls the amount bidder 1 increases the price by in the auction. This can be seen in the new adapted profit functions:

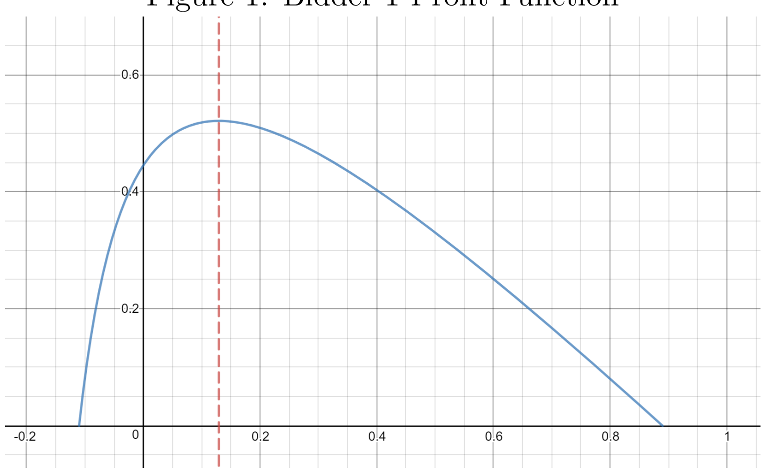
$$\begin{aligned}\hat{\pi}_1(\epsilon) &= (1-w_2-\epsilon)\frac{1}{1+\frac{w_2}{w_2+\epsilon}} \\ &= (1-w_2-\epsilon)\frac{w_2+\epsilon}{2w_2+\epsilon}\end{aligned} \tag{6}$$

$$\begin{aligned}\hat{\pi}_2(\epsilon) &= (1-w_2-\epsilon)\frac{\frac{w_2}{w_2+\epsilon}}{1+\frac{w_2}{w_2+\epsilon}} \\ &= (1-w_2-\epsilon)\frac{w_2}{2w_2+\epsilon}\end{aligned} \tag{7}$$

The new price is $p^*_{new} = w_2 + \hat{\epsilon} \geq w_2$, where $\hat{\epsilon} = \max(\epsilon, 0)$. This is because bidder 1 can only increase the price and cannot decrease as bidder 2 would win the auction and claim all of the company. Bidder 1 now picks ϵ such that it maximises the profit function (6) i.e. $\epsilon^* = \arg\max_{\epsilon}(\hat{\pi}_1)$. This can be calculated by taking the partial derivative of π_1 w.r.t ϵ and setting the following equation to zero, see Appendix A on Page 42 for the calculations, whereby we arrive at the equation:

$$\epsilon^* = \sqrt{w_2(w_2+1)} - 2w_2 \tag{8}$$

Figure 1: Bidder 1 Profit Function



Choice Variable on x-axis, Profit on y-axis

Under further analysis, we see that this equation is nonnegative for values of $w_2 \leq \frac{1}{3}$. This means that bidder 1 has an incentive to set a price above w_2, when $w_2 \leq \frac{1}{3}$, in order to capture a larger percentage of the company, albeit for a higher price per unit. Given this equation for the selection of ϵ, we can clearly see the percentage bidder 1 wins when $w_2 \leq \frac{1}{3}$ can be written as:

$$q_1(w_2) = 1 - \sqrt{\frac{w_2}{w_2 + 1}} \tag{9}$$

and from inspection or further calculation, it is clear that bidder 2 walks away from the auction with:

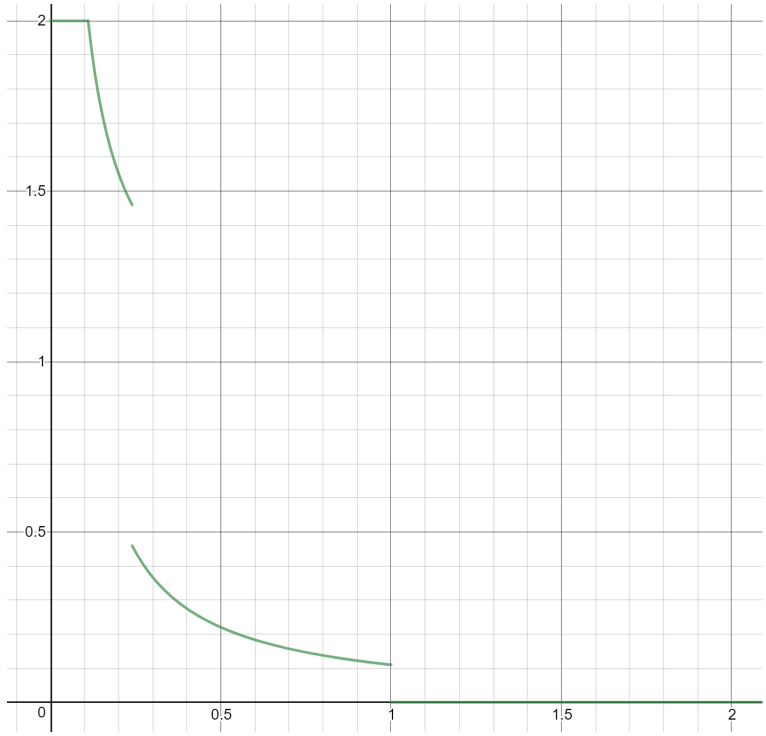
$$q_2(w_2) = \sqrt{\frac{w_2}{w_2 + 1}} \tag{10}$$

due to the fact bidder 2's best response is to always bid their entire budget constraint as long as the price is less than 1, dependent on the firm being good of course. Summing all of these points, we conclude the final demand schedules in the game between two players are:

$$x_1(p) = \begin{cases} 1 & 0 \leq p \leq \min(w_1, w_2 + \hat{\epsilon}) \\ 0 & p > \min(w_1, w_2 + \hat{\epsilon}) \end{cases} \tag{11}$$

$$x_2(p) = \begin{cases} 1 & 0 \leq p \leq w_2 \\ \frac{w_2}{p} & w_2 < p \leq 1 \\ 0 & p > 1 \end{cases} \tag{12}$$

Figure 2: Two Bidder Model: Aggregate Demand



Price on x-axis, Aggregate Demand on y-axis

This concludes a Nash Equilibrium as bidder 2 is submitting a demand schedule (12) which uses their entire budget constraint for all $p \leq 1$ and bidder 1 submits a demand schedule which is a somewhat best response to finding out bidder 2's budget constraint and the worth of the company. The nature of the aggregate demand for the bidders can be seen in Figure 2. Therefore, we have found the optimal strategy for both bidders in this IPO auction. The resulting profit can be broken down into three main events which can be seen below:

$$\pi_1 = \begin{cases} (1 + w_2 - \sqrt{w_2(w_2+1)})(1 - \sqrt{\frac{w_2}{w_2+1}}) & w_2 \in [0, \frac{1}{3}] \text{ and } w_1 \geq \sqrt{w_2(w_2+1)} - 2w_2 \\ (1 - w_1)(\frac{w_1}{w_1+w_2}) & w_2 \in [0, \frac{1}{3}] \text{ and } w_1 < \sqrt{w_2(w_2+1)} - 2w_2 \\ \frac{1-w_2}{2} & w_2 \in (\frac{1}{3}, 1] \end{cases} \tag{13}$$

$$\pi_2 = \begin{cases} (1 + w_2 - \sqrt{w_2(w_2+1)})(\sqrt{\frac{w_2}{w_2+1}}) & w_2 \in [0, \frac{1}{3}] \text{ and } w_1 \geq \sqrt{w_2(w_2+1)} - 2w_2 \\ (1 - w_1)(\frac{w_2}{w_1+w_2}) & w_2 \in [0, \frac{1}{3}] \text{ and } w_1 < \sqrt{w_2(w_2+1)} - 2w_2 \\ \frac{1-w_2}{2} & w_2 \in (\frac{1}{3}, 1] \end{cases} \tag{14}$$

The first case corresponds to the event that bidder 2 is weak enough in terms of budget and bidder 1 is strong enough to set the price to $w_2 + \epsilon^*$ and reap the rewards of doing so.

The second case, on the other hand, is when bidder 2 is weak enough still but bidder 1 is not strong enough to set the price to $w_2 + \epsilon^*$ but instead sets the price to w_1, in which they can just afford the whole company outright, thus still receiving a larger proportion of the company in comparison to bidder 2. The third and final case is the event where bidder 2 does have a budget constraint greater than one third, which is when $\epsilon^* < 0$, in which case the price is set at w_2.

1.6 Three Bidder Model

After studying the two bidder IPO auction, we can expand our initial conditions by incorporating a third bidder, thus increasing the competition in the auction. Again, we make the assumption that $w_1 \geq w_2 \geq w_3$. Using similar analysis to solve the two bidder case, we can construct the profit functions for the three bidders. Similar to the behaviour of bidder 2 in the previous model between two bidders, bidder 2 and 3 submit demand schedules which maximise their allocation for every price p, constrained only by their budget. This is because each bidder knows with certainty the value of the company so in turn demands their maximum for every price $p \leq 1$.

The general profit function must be adapted to incorporate the additional player. Of course, we have an updated aggregate demand, $X(p)$, which sums three demand schedules for every price p. For mostly notation, I introduce $\mathcal{C}_2$, the set of two competitors faced by bidder 1. However, we must adapt the minimum price bidder 1 can set in the auction to integrate bidder 3's demand. In the two bidder case, bidder 1 could force the price as low as w_2, but no lower as bidder 2 would win the auction outright. Therefore, in the three bidder model, if bidder 1 sets the price to either w_2 or w_3, due to the existence of another bidder, they could win the auction for price p greater than w_2 and split the company among themselves according to the allocation rule. The question we ask ourselves is: can bidder 1 still set a price in the three bidder auction? The answer is yes because we can consider bidder 1 competing against one bidder instead of two with budget constraint $w_2 + w_3$. Therefore, following the same logic as before, we know that bidder 1 can set the price as low as $w_2 + w_3$ but due to the strength of the opposition, but no lower.

Now that we know this, we can analyse bidder 1's incentives in this new game among three players, adapting the procedure from before. We do this by introducing the same choice variable ϵ, which captures bidder 1's incentives when faced by different competing budgets. We maximise bidder 1's profit function, (15) and solve for ϵ as so:

$$\pi_1(\epsilon) = (1 - w_2 - w_3 - \epsilon)\left(\frac{1}{1 + \frac{w_2+w_3}{w_2+w_3+\epsilon}}\right) \tag{15}$$

$$= (1 - w_2 - w_3 - \epsilon)\left(\frac{w_2 + w_3 + \epsilon}{2w_2 + 2w_3 + \epsilon}\right) \tag{16}$$

The corresponding partial derivative w.r.t ϵ is:

$$\frac{\partial \pi_1}{\partial \epsilon} = \frac{-3w_2^2 - 3w_3^2 - 6w_2w_3 - 4w_2\epsilon - 4w_3\epsilon + w_2 + w_3 - \epsilon^2}{(2w_2 + 2w_3 + \epsilon)^2} \tag{17}$$

If we set (17) to zero and solve for ϵ, we get the equation below:

$$\epsilon^* = -2w_2 - 2w_3 \pm \sqrt{w_2^2 + w_3^2 + 2w_2w_3 + w_2 + w_3} \tag{18}$$

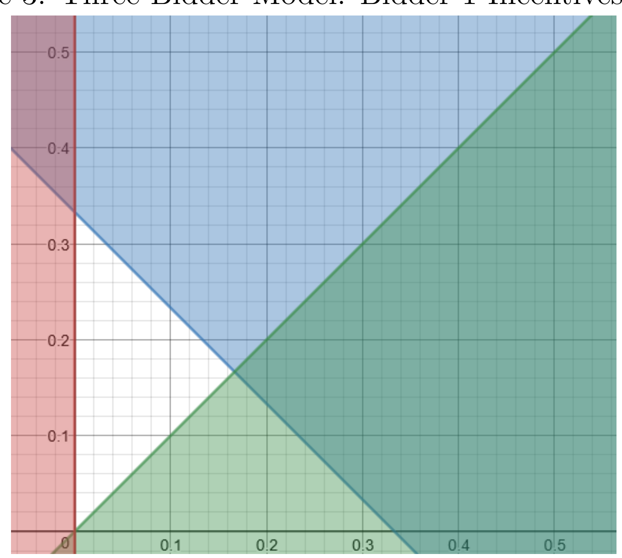
where we take the positive element of the pair of solutions to (17) because a negative ϵ is dominated by the incentives of $\mathcal{C}_2$ so we require $\epsilon \geq 0$. Furthermore, we know for a fact this solution exists because $w_2, w_3 \in [0, 1]$, therefore $w_2^2 + w_3^2 + 2w_2w_3 + w_2 + w_3$ is always nonnegative. It is notable that when $w_3 = 0$, the equations (15) to (18), reduce to the equivalent equations found in this previous model of two bidders in the complete information setting.

If we consider what has been laid out thus far, we can find an intuitive answer to the problem before delving into the mathematics behind the problem. Since, as mentioned previous, we can consider bidder 1 facing one strong player instead of two weaker players, we can see from our analysis in the two player game that bidder 1 has an incentive to increase the price of the auction if $w_2 + w_3 \leq \frac{1}{3}$, otherwise, bidder 1 will set the price at $\min(1, w_2 + w_3)$.

Using fairly simple algebra, we can show this mathematically by setting equation (18) to zero which can be found in Appendix A. The region $\mathcal{R}$ in which bidder 1 has an incentive to increase the market clearing price in the auction is identified in Figure 3, where the region $\mathcal{R}$ is defined below.

$$\mathcal{R} := \left\{ \underbrace{w_2 \leq \frac{1}{3} - w_3,}_{\text{Blue Boundary}} \; \underbrace{w_3 \geq 0}_{\text{Red Boundary}} \; , \; \underbrace{w_2 \geq w_3}_{\text{Green Boundary}} \right\}$$

Figure 3: Three Bidder Model: Bidder 1 Incentives $\epsilon > 0$



w_3 on x-axis, w_2 on y-axis

Of course, in general, the competing bids do not always fall within the region $\mathcal{R}$, in which case, it is in bidder 1's interest to set the price at the minimum of w_2+w_3 and 1. The corresponding demand schedules for the three bidders are as follows:

$$x_1(p) = \begin{cases} 1 & 0 \leq p \leq \min(w_2+w_3+\hat{\epsilon}^*, w_1) \\ \min(\frac{w_1}{p}, 1) & \min(w_2+w_3+\hat{\epsilon}^*, w_1) < p \leq \\ \min(w_2+w_3+\hat{\epsilon}^*, 1) & \\ 0 & p > \min(w_2+w_3+\hat{\epsilon}^*, 1) \end{cases} \tag{19}$$

$$x_{i\neq 1}(p) = \begin{cases} 1 & 0 \leq p \leq w_i \\ \frac{w_i}{p} & w_i < p \leq 1 \\ 0 & p > 1 \end{cases} \tag{20}$$

Comparing demand schedules (19) and (20) to the demand schedules defined in the two bidder model, we can see a clear similarity. The only difference between them is the price bidder 1 changes demand from 1 to 0, which takes into consideration the combined power of the competing bidders. The resulting quantity allocated to each bidder is:

$$q_1(w) = \begin{cases} 1 - \sqrt{\frac{w_2+w_3}{w_2+w_3+1}} \geq 0.5 & (w_3, w_2) \in \mathcal{R} \text{ and } w_1 \geq w_2+w_3+\hat{\epsilon}^* \\ \frac{w_1}{w_1+w_2+w_3} & (w_3, w_2) \in \mathcal{R} \text{ and } w_1 \leq w_2+w_3+\hat{\epsilon}^* \\ \min(\frac{1}{2}, \frac{w_1}{w_1+w_2+w_3}) & (w_3, w_2) \in [0,1]^2 \setminus \mathcal{R} \end{cases} \tag{21}$$

$$q_{i\neq 1}(w) = \begin{cases} \frac{w_i}{\sqrt{(w_2+w_3)(w_2+w_3+1)}} & (w_3, w_2) \in \mathcal{R} \text{ and } w_1 \geq w_2+w_3+\hat{\epsilon}^* \\ \frac{w_i}{w_1+w_2+w_3} & (w_3, w_2) \in \mathcal{R} \text{ and } w_1 \leq w_2+w_3+\hat{\epsilon}^* \\ \frac{1}{2} - \frac{w_2}{w_2+w_3} & (w_3, w_2) \in [0,1]^2 \setminus \mathcal{R} \text{ and } w_1 \geq w_2+w_3 \\ \frac{w_i}{w_1+w_2+w_3} & (w_3, w_2) \in [0,1]^2 \setminus \mathcal{R} \text{ and } w_1 < w_2+w_3 \end{cases} \tag{22}$$

$$p^*(w_2, w_3) = \begin{cases} w_2+w_3+\epsilon^* \leq \frac{1}{3} & (w_3, w_2) \in \mathcal{R} \text{ and } w_1 \geq w_2+w_3+\epsilon^* \\ w_1 & (w_3, w_2) \in \mathcal{R} \text{ and } w_1 \leq w_2+w_3+\epsilon^* \\ \min(1, w_2+w_3) & (w_3, w_2) \in [0,1]^2 \setminus \mathcal{R} \end{cases} \tag{23}$$

The resulting profit for each bidder in each of the three scenarios is thus:

$$\pi_i = (1 - p^*(w_2, w_3))q_i(w_2, w_3) \tag{24}$$

where we use the functions derived above and the pre-determined value of ϵ^* on realisation of competing budget constraints according to (18).

1.7 n Bidder Model

The last phase to analyse to fully understand the IPO auction with complete information is to generalise the number of bidders in the auction. Therefore, we extend the auction to include n bidders competing over the company in the IPO with n different budget constraints, where $w_1 \geq w_2 \geq ... \geq w_{n-1} \geq w_n$. Again, analysing bidder 1's strategy in this auction requires a bit of calculus to derive a function which captures his incentives when faced by n budget constraints. The profit function which incorporates the fact there are now n bidders vying for a proportion of the IPO company is:

$$\pi_1(\epsilon) = (1 - \sum_{i=2}^{n} w_i - \epsilon)\left(\frac{\sum_{i=2}^{n} w_i + \epsilon}{2\sum_{i=2}^{n} w_i + \epsilon}\right) \tag{25}$$

Applying the quotient rule and manipulating the algebra we receive the solutions below:

$$\epsilon^* = -2\sum_{i=2}^{n} w_i \pm \sqrt{\sum_{i=2}^{n} w_i(1 + \sum_{i=2}^{n} w_i)} \tag{26}$$

where we ignore the negative solution, as we did before, and set equal to zero to find the conditions required for equation (26) to be nonnegative. It then comes as no surprise that after doing this calculation, which can be found in Appendix A, we receive the necessary condition:

$$\sum_{i=2}^{n} w_i \leq \frac{1}{3} \tag{27}$$

As a sanity check, we see this is the same condition for both $n = 2$ and $n = 3$, thus the sum of competing wealth must be less than one third in any auction for the wealthiest player to have an incentive to increase the market clearing price in equilibrium, otherwise, the selling price of the company is the minimum of one - the fair selling price in the auction due to the complete information aspect of the auction - and the sum of the competing wealth facing bidder 1.

If each potential buyer in the auction is allocated a budget constraint according to a uniform distribution on $[0, 1]$, then as $n \to \infty$, the probability that the sum of all competing budgets is less than one third tends to zero. This can be seen, without loss of generality, if we let each bidder in $\mathcal{C}_{n-1}$ have the same budget constraint of $B_n = \frac{1}{3(n-1)}$. Then $\mathbb{P}(B_n) = \frac{1}{3(n-1)}$, since we are dealing with a uniform distribution, which gets infinitely small as n approaches infinity.

Without much additional work, we can find the n demand schedules which create a Nash equilibrium in this auction format. The demand schedule for bidder 1 can be recognized as a descendant of the ones established in section 1.5 and 1.6, in particular in equation (11):

$$x_1(p) = \begin{cases} 1 & 0 \leq p \leq \min(\sum_{i=2}^{n} w_i + \hat{\epsilon}^*, w_1) \\ \min(1, \frac{w_1}{p}) & \min(\sum_{i=2}^{n} w_i + \hat{\epsilon}^*, w_1) < p \leq \min(\sum_{i=2}^{n} w_i + \hat{\epsilon}^*, 1) \\ 0 & p > \min(\sum_{i=2}^{n} w_i + \hat{\epsilon}^*, 1) \end{cases} \tag{28}$$

The demand schedules facing bidder 1 are identical as far as strategy goes, and are identical to those seen in previous sections, where they demand the whole company up to their budget constraint then as much as they can get for price less than 1 and zero thereafter.

$$x_{i \neq 1}(p) = \begin{cases} 1 & 0 \leq p \leq w_i \\ \frac{w_i}{p} & w_i < p \leq 1 \\ 0 & p > 1 \end{cases} \tag{29}$$

The demand schedules (28) and (29) form a Nash equilibrium as each bidder has no incentive to deviate as doing so would reduce their profit in the auction by either increasing the price in terms of bidder 1 or reducing the quantity they are allocated by the seller. The aggregate demand for each price p in the n bidder model can be calculated by adding the demands vertically due to the fact they submit their inverse demand functions as so:

$$X(p) = \begin{cases} n & 0 \leq p \leq w_n \\ n - 1 + \frac{w_n}{p} & w_n < p \leq w_{n-1} \\ n - 2 + \frac{w_n + w_{n-1}}{p} & w_{n-1} < p \leq w_{n-2} \\ \vdots & \vdots \\ 1 + \frac{\sum_{i=2}^{n} w_i}{p} & w_2 < p \leq \min(w_1, \sum_{i=2}^{n} w_i + \hat{\epsilon}^*) \\ \min(1, \frac{w_1}{p}) + \frac{\sum_{i=2}^{n} w_i}{p} & \min(w_1, \sum_{i=2}^{n} w_i + \hat{\epsilon}^*) < p \leq \min(\sum_{i=2}^{n} w_i + \hat{\epsilon}^*, 1) \\ \frac{\sum_{i=2}^{n} w_i}{p} & \min(\sum_{i=2}^{n} w_i + \hat{\epsilon}^*, 1) < p \leq 1 \\ 0 & p > 1 \end{cases} \tag{30}$$

2 Analysis

2.1 Budget Constraint Implications

The complete information model identified in the previous section relies heavily on a bidder's budget constraint which was left exogenous. However, since it plays such a major part in the outcome of the IPO auction from sections 1.5 to 1.7, it is fitting that we add some context to where the budget constraint is derived from. The budget constraint incorporates a number of factors which restrict the actions of the bidders. Firstly, the budget constraint takes into account the wealth available to the bidder on commencement of the auction. This wealth is easily accessible and means the payment to the seller can be made immediately. The second main factor is the current portfolio of the bidder. An important aspect to portfolio management is keeping a diversified set of stocks in your portfolio therefore capturing growth from all sectors of an economy and reducing unsystematic risk. Consequently, they do not want to be too exposed in one sector for a high price but are willing to take a large stake in a company if the price is indeed low enough. In addition to this, many laws have been passed globally which restrict asset managers and hedge funds etc. owning large shares of a sector through fear of monopolistic pricing which would damage society. Accordingly, the IPO of a newly founded industry in an economy can be very lucrative for the first company to do so. This can be seen in reality as Uber and Lyft have recently competed against each other to go public first, with the latter winning the race by going public on March 29^{th} of this year. In return, the San Francisco based company raised \$2.34 billion in its IPO, higher than their increased target of $2.2 billion set only a few days prior to Lyft's offering to the NASDAQ stock exchange. However, since its IPO, Lyft's stock price has fallen well below its initial offering price of $72/each to $60 [4].

2.2 Wealth Allocation

Even though this model makes a strong assumption of complete information, we do learn some interesting implications from the model which are replicated empirically. From our previous analysis, it highlights the importance of wealth available in the economy for the IPO auction. We can further extend our analysis by letting total wealth available be exogenous and derive what allocation of that wealth is optimal for the seller. If we let ω be the total available budget for the auction and we consider the two bidder case, where it is clear from our analysis that the wealthier player wants to set the price at $w_2 + \hat{\epsilon}^*$. Therefore, an interesting facet to explore is when an almost monopsonist is better for the seller over equality amongst bidders. In order to do this, we must derive a function for price given wealth allocation.

We know bidder 1's first thought is to set the price to w_2, which for the sake of our analysis will be set to $\omega - w_1$. However, as we have found, when $w_2 \leq \frac{1}{3}$, bidder 1 has an incentive to increase the price of the auction to increase their allocation of the good company in the auction. That said, we add

$$\hat{\epsilon}^* = \max(\sqrt{(\omega - w_1)(\omega - w_1 + 1)} - 2(\omega - w_1), 0)$$

to $\omega - w_1$. It is important to state, by definition of bidder 1's budget constraint, $w_1 \in [\frac{\omega}{2}, \min(1, \omega)]$, due to the fact $w_1 \geq w_2$. However, we know that bidder 1 cannot always afford this price so we must take the minimum of $\hat{\epsilon}^*$ and $2w_1 - \omega$, the latter of which sets the price to w_1 in the case they cannot afford to add $\hat{\epsilon}^*$ to the price and still demand 100% of the company. Therefore, we get:

$$p^* = (\omega - w_1) + \min(2w_1 - \omega, \max(\sqrt{(\omega - w_1)(\omega - w_1 + 1)} - 2(\omega - w_1), 0)) \tag{31}$$

The above equation which is plotted in figures 4 and 5 on Page 19, shows there is a maximum on the interval that w_1 is valid. When $\omega > \frac{2}{3}$, the seller wants perfect equality where $w_1 = w_2 = \frac{\omega}{2}$. However, when $\omega \leq \frac{2}{3}$, the seller prefers to allocate more wealth to one of the bidders in the economy in order to force the price of the auction up. The optimal allocation to bidder 1 can be calculated from the exogenous ω as:

$$w_1 = \omega + \frac{1}{2} - \sqrt{\omega^2 + \frac{1}{4}} \tag{32}$$

Therefore, as $\omega \to 0$, the percentage allocated to bidder 1 tends to 100%. Using the analysis above and from generalizing our model from a two bidder auction to an n bidder auction, we can justify the assumption that when the wealth of the economy is less than $\frac{2}{3}$, the seller prefers to have a monopsonist and splits the rest amongst the $n - 1$ bidders.

Otherwise, the seller prefers perfect equality where no bidder has the power to set the price at some value less than their budget constraint.

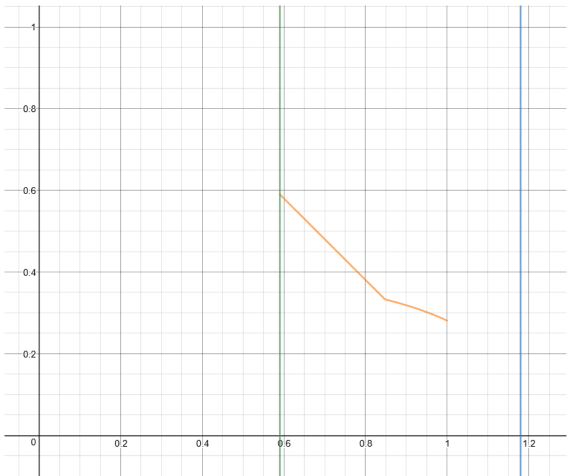


Figure 4: When $\omega > \frac{2}{3}$

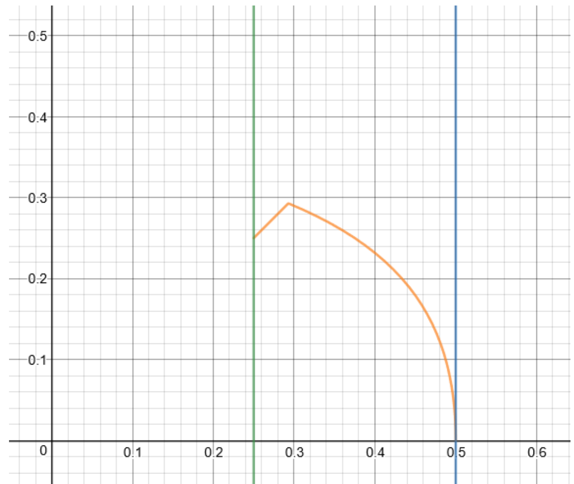


Figure 5: When $\omega \leq \frac{2}{3}$

2.3 Auction Timing

If the bidder observes the wealth of each individual before starting the auction, they can decide whether to open the auction or delay it. Therefore, by observing past and current wealth, they can decide whether the potential bidders are currently experiencing a high or low budget constraint. If the seller notices the bidders have a low budget constraint with respect to past budget constraints, they may forecast their budget constraint to rise in the coming periods, so may delay the auction in order to maximise their wealth in the case the company is indeed a good company. On the other hand, if the seller predicts the bidders have reached a peak in time, they will go ahead with the auction in order to maximise their total revenue. Combining these thoughts with our optimal allocation analysis, we can see in IPO auctions, when the economy is doing well, the seller prefers perfect equality to boost the selling price but when the economy is experiencing a recession or low growth, the seller prefers the wealth to be distributed unevenly to create a monopsonist who owns a larger percentage of total available wealth or to delay the IPO auction to a later date when the economy is in better health.

2.4 Comparison to History

If we take these findings and compare them to real world behaviour, we certainly see more IPOs during years of prosperous growth as can be seen in the figures below, compared to those in years of recession.

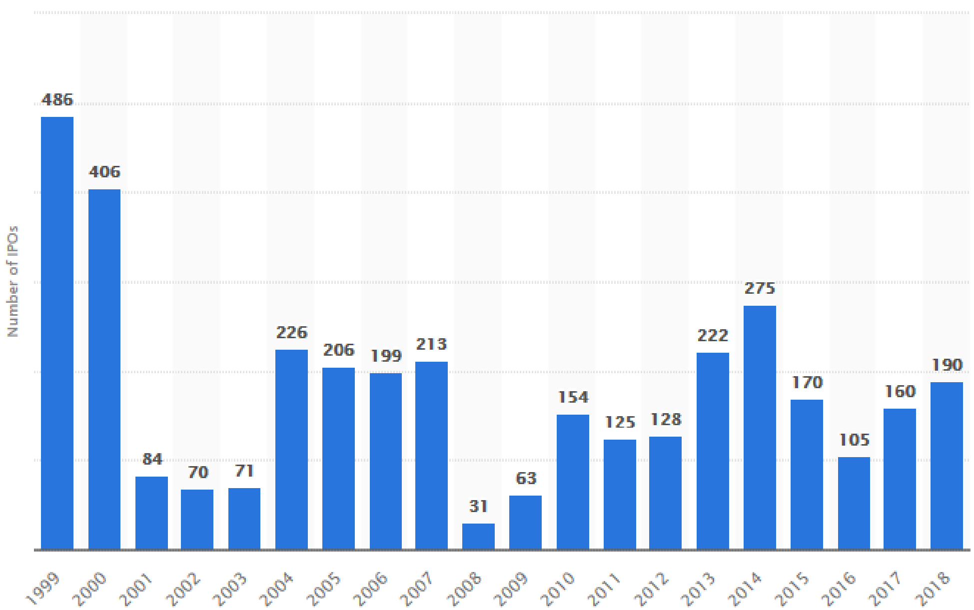


Figure 6: Initial Public Offerings in the US from 1999 to 2018 [13]

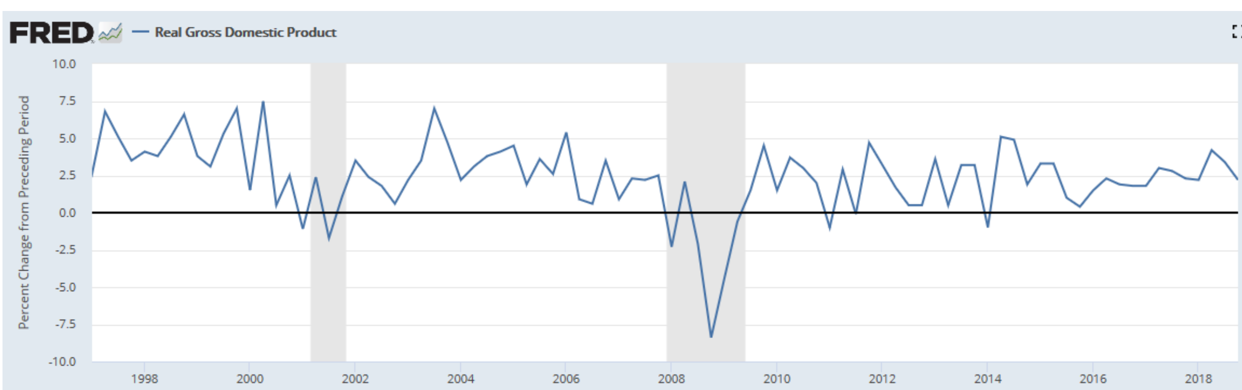
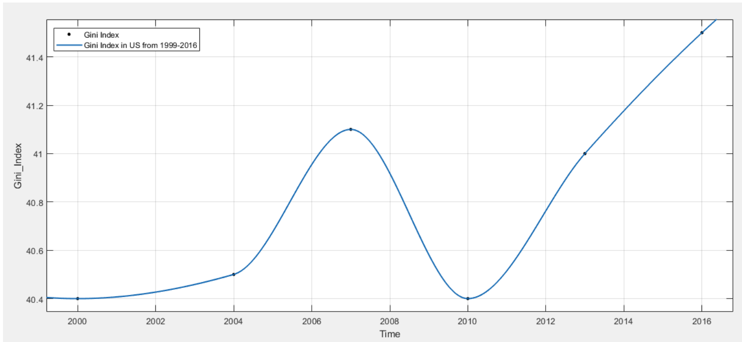


Figure 7: Real GDP Growth in the US from 1997 to 2018 [10]

In particular, when there was a tech bubble during the turn of the century, we saw high GDP growth of around 7.5% which was complemented by extremely high numbers of IPOs in the US, where there were 892 IPOs in two years. Comparing this to the average number of IPOs from 1999 to 2018, which stood at 179.2 IPOs per year, we can clearly see a correlation between economic output and number of IPOs. Moreover, during recessions i.e. after the tech bubble burst and the financial crisis of 2007-08, we see number of IPOs hit a rock bottom, where only 319 IPOs took place in the five corresponding years. Also, we find high levels of growth causes more equality as shown by Figure 8 which illustrates the Gini coefficient since 2000 to 2016 in the US.

Figure 8: Gini Index Interpolated in Time for US in 2000-2016 [2]



Comparing this to our figure on GDP growth, we see equality is improved during times of high economic growth. This gives good economic conditions for the seller as the bidders are more equal and better off than usual, thus more IPOs happen during this period than in others.

But of course, economic agents do not know with certainty the value of an investment before purchase as there is always some risk involved. Therefore, our next model incorporates uncertainty into the model by removing the element of complete information.

Part II
Incomplete Information

3 Model II

Thus far, we have only considered auctions of complete information, where each bidder knows with certainty the worth of the company. To extend our model, we must remove this aspect of certainty and deal with a game of incomplete information. In this instance, we revert to the case where we have two bidders competing in an auction which follows the same model rules as before in terms of payment and allocation. However, instead of receiving a symmetric public signal that the company is good or bad which in turn determines its value, each bidder receives public information on the probability that the firm is good or bad. The intuition behind this probability may come from background empirical research on IPO auctions by the bidders to determine what fraction of firms that go through an Initial Public Offering turn out to be good firms. Furthermore, each agent competing for the company receives private information of the firm's type through internal analysis of the underlying asset on auction. This of course is only observable to the owner of the information and not to the global economy.

3.1 Public Information

The public signal each bidder receives is symmetric and indicates the probability that a firm at random is good or bad. To maintain generality, we say that the distribution of Good and Bad firms in this economy is determined as follows:

$$\mathbb{P}(Good) = m \tag{33}$$

$$\mathbb{P}(Bad) = 1 - m \tag{34}$$

where of course $m \in [0, 1]$. I reiterate the fact, each bidder knows the distribution of good and bad firms and this number is the same for all parties as it is gathered on the same sample space. Without any further research on the value of the company, each bidder would observe this amount and the strength of the competing players (their budget constraints), to inform their bidding behaviour. Of course, in this instance, each bidder would submit demand schedules up to m because in expectation each bidder would be making a profit, or more concretely, on average they would not make a loss.

3.2 Private Information

Unlike public information, bidders receive this signal privately. Both types of firm give out either a high or low signal according to some distribution. To maintain simplicity, we assume a discrete distribution where a firm either gives a loud high signal indicating it is a good firm or a loud low signal indicating it is a bad firm. This private information could be derived from information observed about the particular company at auction rather than some general firm like the public signal, such as financial statements, industry prospects and leadership, amongst other indicators and measures. We then have the following probabilities:

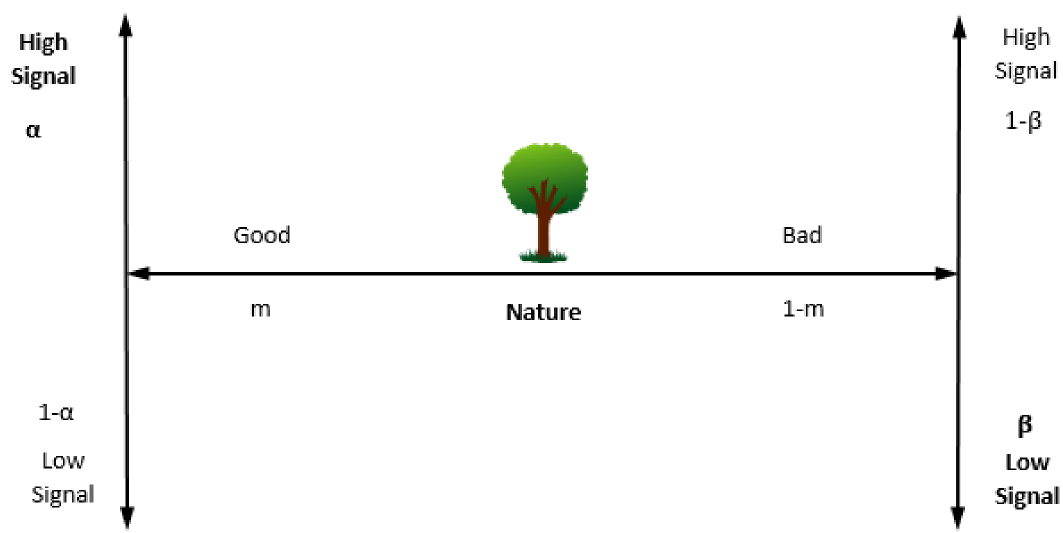
$$\mathbb{P}(\text{High Signal} \mid \text{Good Firm}) = \alpha \tag{35}$$

$$\mathbb{P}(\text{High Signal} \mid \text{Bad Firm}) = 1 - \beta \tag{36}$$

where $\alpha, \beta \in [0, 1]$ and we will assume that $\alpha + \beta > 1$ meaning the bidders more often than not correctly identify a firm's type on average. This distribution is known by all bidders before the auction so they can derive probabilities on the company being good given their signal. We will simplify notation slightly by denoting a high signal as H, a low signal as L, a good firm as G and a bad firm as B. In addition, let $\mathcal{S} = \{H, L\}$ be the set of possible signals bidder i can receive, $\forall i \in \mathcal{B}_n$, *n=number of bidders.*

To more clearly see the set up, Figure 9 illustrates the nature of the public and private information for this game.

Figure 9: Bayesian Belief Revision



3.3 Bayesian Belief Revision

An important aspect of the auction is revising your private beliefs on the type of company at auction. This is done by performing Bayesian belief revision combining your public and private information which will go on to inform your bidding in the auction. Due to the fact α and β are symmetric amongst all bidders in the auction, thus assuming each bidder has the same amount of talent at differentiating between a good and bad firm given background information. This means that none of the bidders are better at deriving the value of the firm over the others. It is therefore the task of the bidders to revise their prior beliefs that a general firm is good, in order to define a probability that the firm being auctioned is good or bad.

To do this we must define two statistics:

- Sensitivity
- Specificity

These statistics identify the probability of assigning a high signal to a good firm and the probability of correctly assigning a low signal to a bad firm. The calculation of these probabilities goes as follows:

$$\begin{aligned} Sensitivity &= \frac{TP}{TP + FN} \\ &= \frac{\alpha}{\alpha + 1 - \alpha} \\ &= \alpha \end{aligned}$$

where TP denotes a true positive which happens when a high signal is assigned to a good firm and FN denotes a false negative which is when a low signal is assigned to a good firm. Therefore, the sensitivity test focuses on the left hand side of figure 9. Our model then identifies the bidders have a sensitivity of $\alpha\%$. Moreover, to calculate the specificity of the bidders, we do a similar calculation:

$$\begin{aligned} Specificity &= \frac{TN}{TN + FP} \\ &= \frac{\beta}{\beta + 1 - \beta} \\ &= \beta \end{aligned}$$

where TN is a true negative and FP is a false positive. The specificity test then focuses on the right hand side of figure 9, giving us a specificity of $\beta\%$ for our bidders. The sensitivity and specificity tests in essence identify in each situation, what is the probability the information they have privately researched is correct? Therefore, it is clear, bidders who have greater talent and experience will have a higher α and/or β signifying the fact they are better as correctly identifying a company's type. However, in our model we assume these values to be symmetric. From the statistics found above, we can derive two likelihood ratios:

$$\begin{aligned} LR_+ &= \frac{Sensitivity}{1 - Specificity} \\ &= \frac{\alpha}{1 - \beta} \end{aligned} \qquad \begin{aligned} LR_- &= \frac{1 - Sensitivity}{Specificity} \\ &= \frac{1 - \alpha}{\beta} \end{aligned}$$

Commonly used in medical testing, the above equations are used by physicians and doctors to help rule in or out a patient from having a disease. For our purposes, we can suppose nature places a disease on humanity and any person chosen at random has the disease with probability m. The bidders, or doctor in this example, then carry out some tests to gather further information on the well-being of the patient and revise their beliefs on whether the patient has the disease. The positive likelihood ratio, LR_+, tells us by how much the probability of having the disease increases from a positive test result and the negative likelihood ratio, LR_-, tells us by how much the probability of having the disease decreases from a negative test result. We then carry out a process which converts pre-test probabilities to post-test probabilities which can be found in appendix B [14].

We have now successfully deduced the post-test probabilities, or post-belief probabilities when receiving a high or low signal from the firm being auctioned. We find, when a bidder receives a high signal after analysing the company, their post-beliefs are:

$$\mathbb{P}(G|H) = \frac{m\alpha}{m\alpha + (1-m)(1-\beta)} \tag{37}$$

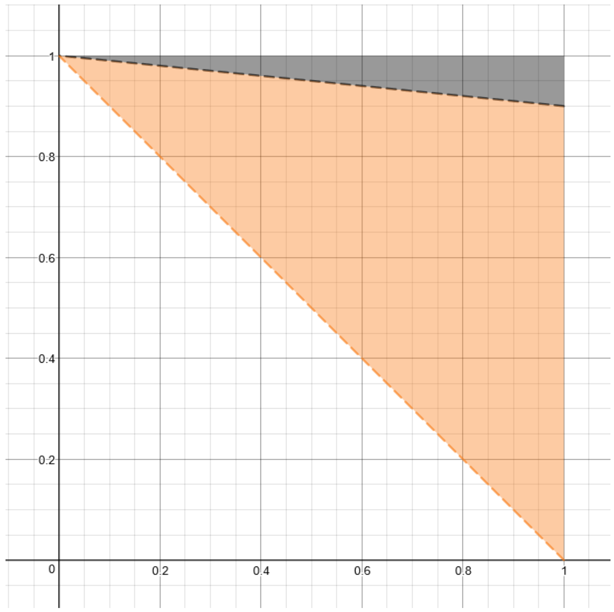
and for the case where the bidder receives a low signal, their post-beliefs are:

$$\mathbb{P}(G|L) = \frac{m(1-\alpha)}{m(1-\alpha) + (1-m)\beta} \tag{38}$$

These probabilities are highly intuitive as the probability of being a good firm given a high signal is the probability of receiving a high signal from the company and that company is good over the probability of receiving a high signal. The other three cases can be similarly explained. From our assumption of $\alpha + \beta > 1$, we find that $\mathbb{P}(G|H) > \mathbb{P}(G|L)$, proof can be found in appendix B, meaning on receipt of a high signal, a bidder knows the probability of being a good firm is higher than if they had received a low signal. Above all else, this makes sense, as on receipt of good news about the value of the company at auction, the bidder has higher expectations on its value over a bidder who receives a low signal prior to the auction.

Looking at the likelihood ratios, we see via our assumption of $\alpha + \beta > 1$ that $LR_+ > 1$ and $LR_- < 1$. This tells us that the researching the company prior to the auction does tell us something about the value of the company. In finance, this argument is between whether active or passive management is beneficial in the long run. If the bidder(s) have likelihood ratios equal to or close to 1, then we can deduce the test does not tell you a lot about the value of the company. In which case, passive management is the optimal strategy for asset managers. A good positive likelihood ratio is considered to be one greater than 10 and a good negative likelihood ratio is one less than 0.1. Therefore, a good LR requires a more restrictive assumption on α and β, where $\alpha + 0.1\beta > 1$ illustrated in the below figure:

Figure 10: Constraints on α and β



(a) α on x-axis and β on y-axis

The black region in figure 10 signifies the constraint $\alpha + 0.1\beta > 1$, where β is plotting on the x-axis and α on the y-axis. In comparison, to the original assumption of $\alpha + \beta > 1$ illustrated by the orange and black region in the above figure, we can easily see the significance between the two assumptions.To have a successful value test, both the specificity and sensitivity is required to be almost perfect which would be very costly for the bidders in real life as this would require a lot of additional training for the research team as well as an oracle on-sight to see into the future.

3.4 Removing the Bidding Constraint

The constraint restricting bidders from submitting demand greater than 1 was dropped for the incomplete case. This means bidders can fully exert their budget constraint for every price p. The reason for the exclusion was because we would discover no further interesting facets from the complete case as bidder 1 would only be interested in increasing the price of the auction when $w_2 \leq \frac{1}{3}\mathbb{E}(V|t_1)$, where t_1 is the information bidder 1 receives. Therefore, it is obvious to see the affiliation between the complete and incomplete case as in the complete case if $t_1 = H$ then $\mathbb{E}(V|t_1) = 1$, otherwise, $\mathbb{E}(V|t_1) = 0$. However, by allowing bidders to demand more than 100% of the company, we discover a whole new method of bidding which uses hedging and removes the winner's curse as we show in Section 3.5.

3.5 Two Bidder Model

The model without certainty over the value of the company, of course, complicates the equilibrium as you can face a player with a high signal or a low signal which didn't happen in the complete case. The complete information case can be thought of as the incomplete information model with $\alpha = 1$ and $\beta = 1$, i.e. the specificity and sensitivity is 100%. In turn, forcing symmetric types in terms of signal as there is no chance of receiving a different signal to what the other received. Using the same allocation rule (1) and payment rule (2) from Page 8, we change the bidding rule by removing the upper restriction for the bidders allowing them to use their entire budget constraint at every price p as reasoned in Section 3.4.

Due to the lack of certainty in this auction, we must calculate the expected value given a bidder's private information. In the two bidder case, the bidders do not just consider the information they receive but taken into consideration the possible type of the other bidder. We must calculate the probability of being a good player given my type, t_i and the type of the competing bidder, t_j.

In order to calculate this probability, we use Bayes' Theorem (39).

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}[3] \tag{39}$$

where $A = G$ and $B = t_i \cap t_j$ as so:

$$\mathbb{P}(G|t_i, t_j) = \frac{\mathbb{P}(G \cap t_i \cap t_j)}{\mathbb{P}(t_i \cap t_j)} \text{ for } t_i, t_j \in \mathcal{S}, i \neq j \tag{40}$$

where $\cap$ denotes the intersection of events so $\mathbb{P}(G \cap t_i \cap t_j)$ means the probability of a good firm and the two types of the bidders are t_i and t_j. However, equation (40) leaves two probabilities to calculate. The first of those is $\mathbb{P}(t_i \cap t_j)$.

$$\mathbb{P}(t_i \cap t_j) = m\mathbb{P}(t_i|G)\mathbb{P}(t_j|G) + (1-m)\mathbb{P}(t_i|B)\mathbb{P}(t_j|B) \tag{41}$$

The second equation can be taken straight from equation (41), my taking the segment where the company is good. This gives:

$$\mathbb{P}(G \cap t_i \cap t_j) = m\mathbb{P}(t_i|G)\mathbb{P}(t_j|G) \tag{42}$$

Therefore, combining equations (41) and (42), we get the desired equation for $\mathbb{P}(G|t_i, t_j)$. The four possible combinations of type between the two bidders are:

$$\mathbb{P}(G|H,H) = \frac{m\alpha^2}{m\alpha^2 + (1-m)(1-\beta)^2} \qquad \mathbb{P}(G|H,L) = \frac{m\alpha(1-\alpha)}{m\alpha(1-\alpha) + (1-m)\beta(1-\beta)}$$
$$\mathbb{P}(G|L,H) = \frac{m\alpha(1-\alpha)}{m\alpha(1-\alpha) + (1-m)\beta(1-\beta)} \quad \mathbb{P}(G|L,L) = \frac{m(1-\alpha)^2}{m(1-\alpha)^2 + (1-m)\beta^2} \tag{43}$$

In our model, the expected values are easy to calculate due to the fact, if the company is good then it is worth £1 and nothing if a bad firm. The expected values are then:

$$\mathbb{E}(V|H,H) = \frac{m\alpha^2}{m\alpha^2 + (1-m)(1-\beta)^2} \qquad \mathbb{E}(V|H,L) = \frac{m\alpha(1-\alpha)}{m\alpha(1-\alpha) + (1-m)\beta(1-\beta)}$$
$$\mathbb{E}(V|L,H) = \frac{m\alpha(1-\alpha)}{m\alpha(1-\alpha) + (1-m)\beta(1-\beta)} \qquad \mathbb{E}(V|L,L) = \frac{m(1-\alpha)^2}{m(1-\alpha)^2 + (1-m)\beta^2}$$

The final equation we must define is $\mathbb{P}(t_i|t_j)$ which is found via Bayes' Theorem. The probability of the intersection of t_i and t_j is shown in equation (41), and the probability of receiving t_i for bidder i is $m\mathbb{P}(t_i|G) + (1-m)\mathbb{P}(t_i|B)$. This gives:

$$\mathbb{P}(H|H) = \frac{m\alpha^2 + (1-m)(1-\beta)^2}{m\alpha + (1-m)(1-\beta)} \qquad \mathbb{P}(L|L) = \frac{m(1-\alpha)^2}{m(1-\alpha)^2 + (1-m)\beta^2} \tag{44}$$

where the $\mathbb{P}(L|H)$ and $\mathbb{P}(H|L)$ can be found by subtracting the corresponding equations in (44) as they are the complements. Now that we have identified the necessary equations we can continue to the derivation of strategies for both bidders in this IPO auction.

3.5.1 Strategy Derivation

Unlike the complete case where the bidders knew exactly the value of the company so demanded as much as they could get for any price up to its true value, the bidders must now consider the impact of the winner's curse. We take a step-by-step derivation of the bidders' strategies under each possible scenario, focusing on different strategies given different wealth levels. We must identify a players' strategy in four different wealth intervals and their permutations:

- $[0, \mathbb{E}(V|L,L)]$
- $(\mathbb{E}(V|L,L), \mathbb{E}(V|H,L)]$
- $(\mathbb{E}(V|H,L), \mathbb{E}(V|H,H)]$
- $(\mathbb{E}(V|H,H), 1]$

The first of these is when $w_1, w_2 \in [0, \mathbb{E}(V|L,L)]$, which is very similar to that of the complete case as no matter what your and the opposing player's signal, you know the company in expectation is at least $\mathbb{E}(V|L,L)$, which is the expected value of the company in the event both players receive a low signal from their research. We adopt the same assumption as before, where $w_1 \geq w_2$ and bidder 1 acts as a price setter in the auction. However, due to the removal of an upper bound on quantity demanded, bidder 1 no longer has an incentive to increase the market clearing price as they can use their wealth advantage in the auction to increase the quantity they are allocated instead of increasing the price to do the same job. Thus, bidder 1 demands zero units of the company for prices greater than w_2, no matter the signal they receive. Furthermore, due to fact the minimum expected value is greater than w_2, bidder 1 will submit a demand schedule which demands a budget binding quantity up to w_2. As for bidder 2, they also demand a budget binding amount but up to $\mathbb{E}(V|L,L)$

then the minimum of $0.5 - \delta$ and their maximum quantity given their budget constraint up to $\mathbb{E}(V|t_2, H)$ and zero thereafter. But of course, in equilibrium, any demand higher than w_2 does not matter as bidder 1 is setting the price at w_2. Therefore, all together we get the demand schedules below, where we use the notation x_{it_i} to denote the demand of bidder i who receives private information t_i:

$$x_{1t_1} = \begin{cases} \frac{w_1}{p} & p \in [0, w_2] \\ 0 & p \in (w_2, \infty) \end{cases} \tag{45}$$

$$x_{2t_2} = \begin{cases} \frac{w_2}{p} & p \in [0, \mathbb{E}(V|L, L)] \\ \min(0.5 - \delta, \frac{w_2}{p}) & p \in (\mathbb{E}(V|L, L), \mathbb{E}(V|t_2, H)] \\ 0 & p \in (\mathbb{E}(V|t_2, H), \infty) \end{cases} \tag{46}$$

The profit made by both bidders from this auction, given the level of wealth are:

$$\mathbb{E}(\pi_{1t_1}) = \frac{w_1}{w_1 + w_2}\mathbb{P}(L|t_1)(\mathbb{E}(V|t_1, L) - w_2) + \frac{w_1}{w_1 + w_2}\mathbb{P}(H|t_1)(\mathbb{E}(V|t_1, H) - w_2)$$
$$\mathbb{E}(\pi_{2t_2}) = \frac{w_2}{w_1 + w_2}\mathbb{P}(L|t_2)(\mathbb{E}(V|t_2, L) - w_2) + \frac{w_2}{w_1 + w_2}\mathbb{P}(H|t_2)(\mathbb{E}(V|t_2, H) - w_2)$$

The only change bidder 2 could make in this auction to effect the outcome would be to decrease their quantity demanded for any price p, but this will not change the selling price of the firm by construction, due to the fact $w_1 \geq w_2$. The only changes bidder 1 can make is to increase the market clearing price in the auction, in which case their expected profit will monotonically decrease because $\frac{\partial \mathbb{E}(\pi_{1t_1})}{\partial p} < 0$. Otherwise, if they try decrease the market clearing price, they will fail as by construction, bidder 2 demands 1 at $p = w_2$, thus the whole company would go to bidder 2. Therefore, we have a Nash equilibrium as neither bidder can increase their profit by unilaterally deviating from their current position.

The second case we must consider is when the budget constraints belong to the interval $(\mathbb{E}(V|L, L), \mathbb{E}(V|H, L)]$ i.e. $\mathbb{E}(V|L, L) < w_2 \leq w_1 \leq \mathbb{E}(V|H, L)$. Unlike the previous case, the information each bidder receives effects the outcome of the auction. Looking at the previous case, we see no matter what private information the bidders receive, they submit demand which is budget binding and bidder 1 uses their budget advantage to gather more allocation for every price. That said, we now must construct the optimal behaviour for both bidders for $p \in [\mathbb{E}(V|L, L), \mathbb{E}(V|L, H)]$ now that bidder 1 can no longer stop the price below $\mathbb{E}(V|L, L)$. If bidder i receives a low signal, and bidder j receives a low signal as well, the expected value is $\mathbb{E}(V|L, L)$ so they can adopt a conservative approach and demand $0.5 - \delta$, where δ is the smallest increment the bidders can demand. This means that if the other bidder is also a low type and submits a conservative demand, the selling price is $\mathbb{E}(V|L, L)$ as $2(0.5 - \delta) < 1$ so by the construction of the market clearing price this does not set the price. However, if the opposing player receives a high signal then the expected value is $\mathbb{E}(V|H, L)$ but cannot afford to buy the company outright for price $p > \mathbb{E}(V|H, L)$ due to their budget constraint, and of course, they wouldn't want to do this as they would make an expected loss. However, similar to the previous case, the high bidder will demand their maximum amount given their budget constraint up to $\mathbb{E}(V|H,L)$ due to their private information. Therefore, we will analyse the sub-game between the two players:

Table 1: Low Bidder Game

		1	
		S_{1L}^1	S_{1L}^2
2	S_{2L}^1	(**Purple**, *Purple*)	(**Green**, *Green*)
	S_{2L}^2	(**Blue**, *Blue*)	(**Orange**, *Orange*)

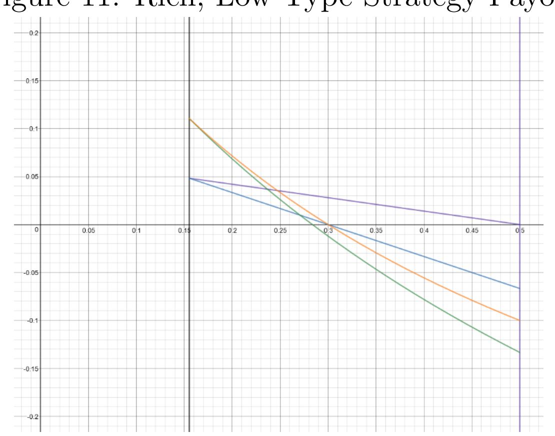
where we use the below definitions:

$$S_{1L}^1 := \begin{cases} \frac{w_1}{p} \text{ for } p \in [0, \mathbb{E}(V|LL)] \\ 0.5 - \delta \text{ for } p \in (\mathbb{E}(V|LL), w_2] \\ 0 \text{ for } p \in (w_2, \infty) \end{cases} \qquad S_{2L}^1 := \begin{cases} \frac{w_2}{p} \text{ for } p \in [0, \mathbb{E}(V|LL)] \\ 0.5 - \delta \text{ for } p \in (\mathbb{E}(V|LL), \mathbb{E}(V|HL)] \\ 0 \text{ for } p \in (\mathbb{E}(V|HL), \infty) \end{cases}$$

$$S_{1L}^2 := \begin{cases} \frac{w_1}{p} \text{ for } p \in [0, w_2] \\ 0 \text{ for } p \in (w_2, \infty) \end{cases} \qquad S_{2L}^2 := \begin{cases} \frac{w_2}{p} \text{ for } p \in [0, \mathbb{E}(V|HL)] \\ 0 \text{ for } p \in (\mathbb{E}(V|HL), \infty) \end{cases}$$

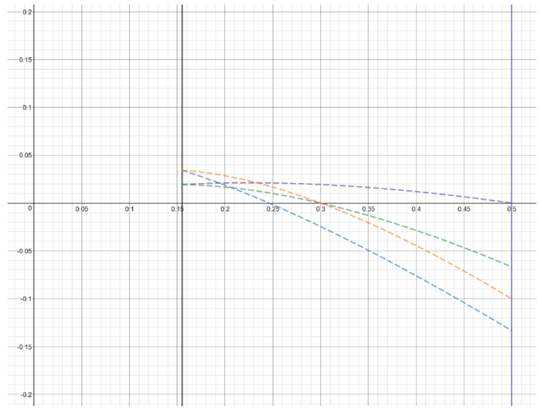
The colours in table 1, are the corresponding outcomes for both bidders which are illustrated in figures 11 and 12 for bidder 1 and 2, respectively. The profit function depends on the strategy you use as well as the competing strategy but the basic premise is the same as equation (4) on page 9. Observing figures 11 and 12, we see it is a Nash equilibrium for both bidders respectively to follow strategies S_{1L}^1 and S_{2L}^1 on the interval $[\phi, \mathbb{E}(V|H,L)]$ where the formula for ϕ is in Appendix C, and denotes the intersection of bidder 1's orange and purple profit equations in Figure 11. However, the bidders should follow strategies S_{1L}^2 and S_{2L}^2 on the interval $(\mathbb{E}(V|L,L), \phi)$ as this is bidder 2's best response to bidder 1.

Figure 11: Rich, Low Type Strategy Payoffs



(a) Budget Constraint of Bidder 2 on x-axis and $\mathbb{E}(\pi)$ for bidder 1 on y-axis

Figure 12: Poor, Low Type Strategy Payoffs



(a) Budget Constraint of Bidder 2 on x-axis and $\mathbb{E}(\pi)$ for bidder 2 on y-axis

The above strategies can be seen from figures 11 and 12, where it is easy to see S^2_{1L} dominates S^1_{1L} on some interval and the reverse on the complement of this interval. Bidder 2 observes this fact and decides between the payoff of green and orange as a best response to bidder 1's incentives. On the corresponding interval, it is clear orange dominates green which happens when bidder 2 plays S^2_{2L}. On the other hand, we see bidder 2 has an incentive to change to S^1_{2L} as they want to switch to the purple payoff, however in doing so, because bidder 1 does not having the same deviation incentive, they would receive the green payoff which is dominated by orange on $(\mathbb{E}(V|L,L),\phi)$, so bidder 2 sticks. However, when $w_2 > \phi$, bidder 1 also has an incentive to deviate to S^1_{1L} which shows a Nash equilibrium. Of course, bidder 1 can still set the price of the auction, and they do this at w_2.

Therefore, the optimal demand schedules for the low types are:

$$x_{1L} = \begin{cases} \frac{w_1}{p} & p \in [0, \mathbb{E}(V|L,L)] \\ \min(0.5-\delta, \frac{w_1}{p}) & p \in (\mathbb{E}(V|L,L), w_2] \text{ if } w_2 \geq \phi \\ \frac{w_1}{p} & p \in (\mathbb{E}(V|L,L), w_2] \text{ if } w_2 < \phi \\ 0 & p \in (\mathbb{E}(V|H,L), \infty) \end{cases} \tag{47}$$

$$x_{2L} = \begin{cases} \frac{w_2}{p} & p \in [0, \mathbb{E}(V|L,L)] \\ \min(0.5-\delta, \frac{w_2}{p}) & p \in (\mathbb{E}(V|L,L), \mathbb{E}(V|H,L)] \text{ if } w_2 \geq \phi \\ \frac{w_2}{p} & p \in (\mathbb{E}(V|L,L), \mathbb{E}(V|H,L)] \text{ if } w_2 < \phi \\ 0 & p \in (\mathbb{E}(V|H,L), \infty) \end{cases} \tag{48}$$

and on receipt of high information, their demand schedules should be:

$$x_{1H} = \begin{cases} \frac{w_1}{p} & p \in [0, w_2] \\ 0 & p \in (w_2, \infty) \end{cases} \tag{49}$$

$$x_{2H} = \begin{cases} \frac{w_2}{p} & p \in [0, \mathbb{E}(V|H,L)] \\ \min(0.5-\delta, \frac{w_2}{p}) & p \in (\mathbb{E}(V|H,L), \mathbb{E}(V|H,H)] \\ 0 & p \in (\mathbb{E}(V|H,H), \infty) \end{cases} \tag{50}$$

where it bidder 2 submits demand on $[w_2, \mathbb{E}(V|H,H)]$ to ensure bidder 1 sets the price at w_2 and doesn't second guess bidder 2 and increase the auction price marginally to capture all of the company.

The third case to be considered is the event when $\mathbb{E}(V|H,L) < w_2 \leq w_1 \leq \mathbb{E}(V|H,H)$. In this case, the low type acts in the same manor as on the interval $[\phi, \mathbb{E}(V|H,L)]$ in the previous case, so the interesting analysis comes from observing the behaviour of the high types in this auction. Due to the fact, a low type doesn't demand any of the company for $p > \mathbb{E}(V|H,L)$ as they would make an expected loss, the high type knows if they are facing competition at this price level, they must be against another high type which means the expected value is $\mathbb{E}(V|H,H)$. Therefore, bidder 1 has a strategic decision to make when submitting their demand schedule as they are still as wealthy as bidder 2. As in the previous case, bidder 1 expects the value of the company to be at least $\mathbb{E}(V|H,L)$, so submit budget binding demand for price below this value. Bidder 2 recognises the same incentives so the bidders do not shade the price as in doing so, recognising the fact the other has enough wealth to purchase the company outright in the case they are a high type. Similar to the previous case, the two bidders must decide what strategy to play in the auction. The bidders decide whether to play conservatively or aggressively. However, unlike the previous case, it doesn't matter what bidder 2 does as bidder 1 sets the price to w_2. Therefore, we focus our attention on bidder 1's strategic decision. They have the choice between:

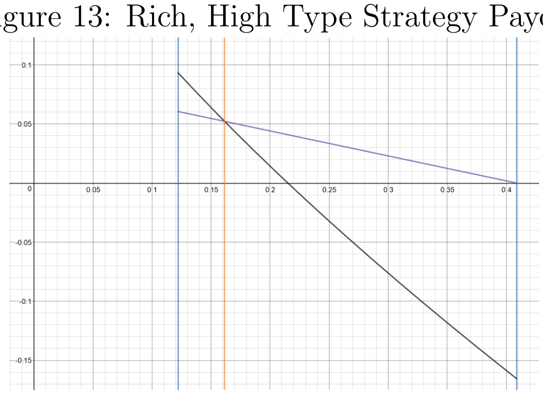
$$S^1_{1H} = 1 - \delta \text{ for } p \in (\mathbb{E}(V|H,L), w_2] \tag{51}$$

and

$$S^2_{1H} = \frac{w_1}{p} \text{ for } p \in [\mathbb{E}(V|H,L), w_2] \tag{52}$$

The profit from picking each strategy is shown in figure 13, where S^1_{1H} is purple and S^2_{1H} is black. The blue lines show the boundaries on the possible values w_2 can take in this scenario. From figure 13, we see that it is beneficial to play S^2_{1H} for low w_2 and S^1_{1H} for higher w_2. The point of deviation happens at point $w_2 = \xi$, the formula for which can be found in appendix C, and is highlighted by the orange vertical line in figure 13.

Figure 13: Rich, High Type Strategy Payoffs



The resulting demand schedules for the two high bidders is:

$$x_{1H} = \begin{cases} \frac{w_1}{p} & p \in [0, \mathbb{E}(V|H,L)] \\ 1-\delta & p \in (\mathbb{E}(V|H,L), w_2] \text{ if } w_2 > \xi \\ \frac{w_1}{p} & p \in (\mathbb{E}(V|H,L), w_2] \text{ if } w_2 \leq \xi \\ 0 & p \in (w_2, \infty) \end{cases} \tag{53}$$

and

$$x_{2H} = \begin{cases} \frac{w_2}{p} & p \in [0, \mathbb{E}(V|H,L)] \\ \min(1-\delta, \frac{w_2}{p}) & p \in (\mathbb{E}(V|H,L), \infty) \end{cases} \tag{54}$$

From the figure above, we see that the expected profit for bidder 1 is always nonnegative, no matter the strength of the opposition.

The final case we must consider is when $\mathbb{E}(V|H,H) < w_2 \leq w_1$. This means that both bidders can afford the buy the company outright for its maximum possible expected value of $\mathbb{E}(V|H,H)$. Again, this doesn't effect a bidder who receives a low informational signal from the firm being auctioned as they will only demand part of the company up to $\mathbb{E}(V|H,L)$. Also, similar to the high type, they only demand a share in the company up to price $\mathbb{E}(V|H,H)$. Therefore, bidder 1 no longer has the capability to set the price of the auction but still benefits from having a higher budget constraint as they can demand a higher quantity than bidder 2 for every price p. Therefore, high types act conservatively and demand $1-\delta$ as so:

$$x_{1H} = \begin{cases} \frac{w_1}{p} & p \in [0, \mathbb{E}(V|H,L)] \\ 1-\delta & p \in (\mathbb{E}(V|H,L), \mathbb{E}(V|H,H)] \\ 0 & p \in (\mathbb{E}(V|H,H), \infty) \end{cases} \tag{55}$$

$$x_{2H} = \begin{cases} \frac{w_2}{p} & p \in [0, \mathbb{E}(V|H,L)] \\ 1-\delta & p \in (\mathbb{E}(V|H,L), \mathbb{E}(V|H,H)] \\ 0 & p \in (\mathbb{E}(V|H,H), \infty) \end{cases} \tag{56}$$

and low types act as expected:

$$x_{1L} = \begin{cases} \frac{w_1}{p} & p \in [0, \mathbb{E}(V|L,L)] \\ 0.5-\delta & p \in (\mathbb{E}(V|L,L), \mathbb{E}(V|H,L)] \\ 0 & p \in (\mathbb{E}(V|H,L), \infty) \end{cases} \tag{57}$$

$$x_{2L} = \begin{cases} \frac{w_2}{p} & p \in [0, \mathbb{E}(V|L,L)] \\ 0.5-\delta & p \in (\mathbb{E}(V|L,L), \mathbb{E}(V|H,L)] \\ 0 & p \in (\mathbb{E}(V|H,L), \infty) \end{cases} \tag{58}$$

The final cases involve the permutations of picking two wealth levels for the bidders from the four intervals:

$[0, \mathbb{E}(V\|LL)]$	$(\mathbb{E}(V\|LL),\mathbb{E}(V\|HL)]$
$(\mathbb{E}(V\|HL),\mathbb{E}(V\|HH)]$	$(\mathbb{E}(V\|HH), 1]$

The cross section of wealth levels can be seen in the tables on pages 35 to 36. The high type uses their superior budget constraint effectively by playing the same strategy as they would if their budget constraint was on the same interval as bidder 2's but in doing so demanding more by fully exerting their budget constraint where applicable.

Table 2: Demand Schedules for Bidders 1 and 2 receiving Low Signals by Wealth Level

Bidder 1 \ Bidder 1	$[0, \mathbb{E}(V\|LL)]$	$(\mathbb{E}(V\|LL), \phi]$	$(\phi, \mathbb{E}(V\|HL)]$	$(\mathbb{E}(V\|HL), \xi]$	$(\xi, \mathbb{E}(V\|HH)]$	$(\mathbb{E}(V\|HH), 1]$
$[0, \mathbb{E}(V\|LL)]$	(S^2_{1L}, S^1_{2L})					
$(\mathbb{E}(V\|LL), \phi]$	-	(S^2_{1L}, S^2_{2L})				
$(\phi, \mathbb{E}(V\|HL)]$	-	-	(S^1_{1L}, S^1_{1L})			
$(\mathbb{E}(V\|HL), \xi]$	-	-	-	(S^1_{1L}, S^1_{2L})		
$(\xi, \mathbb{E}(V\|HH)]$	-	-	-	-	(S^1_{1L}, S^1_{2L})	
$(\mathbb{E}(V\|HH), 1]$	-	-	-	-	-	(S^1_{1L}, S^1_{2L})

Table 3: Demand Schedules for Bidders 1 and 2 receiving High Signals by Wealth Level

Bidder 2 \ Bidder 1	$[0, \mathbb{E}(V\|LL)]$	$(\mathbb{E}(V\|LL), \phi]$	$(\phi, \mathbb{E}(V\|HL)]$	$(\mathbb{E}(V\|HL), \xi]$	$(\xi, \mathbb{E}(V\|HH)]$	$(\mathbb{E}(V\|HH), 1]$
$[0, \mathbb{E}(V\|LL)]$	(S^2_{1H}, S^1_{2H})					
$(\mathbb{E}(V\|LL), \phi]$	-	(S^2_{1H}, S^1_{2H})				
$(\phi, \mathbb{E}(V\|HL)]$	-	-	(S^1_{1H}, S^1_{2H})			
$(\mathbb{E}(V\|HL), \xi]$	-	-	-	(S^1_{1H}, S^1_{2H})	(S^2_{1H}, S^1_{2H})	
$(\xi, \mathbb{E}(V\|HH)]$	-	-	-	-	(S^1_{1H}, S^1_{2H})	
$(\mathbb{E}(V\|HH), 1]$	-	-	-	-	-	(S^1_{1H}, S^1_{2H})

4 Analysis

4.1 Talent Gathering

The main difference between the complete and incomplete case is: alpha and beta are no longer equal to 1, so there is some uncertainty in the investment. This highlights the importance of information in the economy, in particular, the ability of firms to analyse and accurately price stocks. A common argument in finance is that analysts do not have the necessary skills or talent to accurately and persistently price stocks accurately due to the amount of information that needs to be taken into account, therefore, in our model having a low α and/or β.[7]. This would violate our assumption of $\alpha + \beta > 1$, which is so prevalent in Model II. As a consequence, it is beneficial to the seller to stock pile a number of bidders who have a high α and β to accurately price the company, in order to give the company a stable stock price. We see this regularly in the real world as many private companies go to a small group of investment banks to help them go public such as Goldman Sachs and J.P Morgan Chase. In turn, they receive a fee for their support which is reinvested in their recruitment to increase or maintain their talent pool which attracts further business in the future. However, a counter argument can be made that companies only go to these companies because they are distinguished names in the world of finance, so by not going to them when going public may send a negative signal to the economy as they have opted for a second tier underwriter.

4.2 Bidding Behaviour

The incomplete information model captures the behaviour of the bidders at low and high prices well. The model identifies the idea that bidders bid aggressively at low prices as seen in tables 2 and 3 but act more cautiously at higher prices to avoid high losses. They do this by submitting demand schedules which do not fully exert their budget constraint or fully exert if the budget constraint means they cannot demand high amounts of the company in the auction.

4.3 First Day Returns Anomaly

An interesting aspect to Initial Public Offerings alluded to in Section 2.1 is the phenomena of higher end of day first day stock prices to offering price. The example mentioned earlier was about Lyft who managed to raise more than their target from its IPO. However, the end of day stock price was 8.73% higher than its offering price, which for one day is an exceptional ROI. Furthermore, this is not an isolated anomaly as empirically the return is on average 16% [4] [1]. The incomplete information model can suggest a possible reason for this increase.

If we let the auction be between two bidders but be observed by an audience of potential bidders who can bid for the company on conclusion of the auction. The bidders participating in the auction do not know what the information of the opposing player is but when the IPO auction is completed, both realise the information the opponent had, assuming they were playing the Nash equilibrium. In addition to this, not only do each of the players realise the information the opponent had but so do the audience. Therefore, those who also conducted research on the company can re-adjust their beliefs given the realisation of two other bidders' information. If we let an audience member have a high signal and witnesses an auction which had two high types face each other then the probability of the company being good given three high types is obviously higher than the probability of a good firm given two high types. In turn, the expected value is higher. Thus, the audience member can go to one of the two owners with an offer to purchase a portion of the company for a price higher than that paid in the auction thus forcing the price up. This also alludes to the fact that initial public offerings can be favourable to those active in the initial stages as the price can be low due to the lack of information. However, once the company goes public, bidders realize other bidders' signals and re-adjust their beliefs with only bidders who receive high signals approaching the owners and the low signal types stay away as their expected value is below $\mathbb{E}(V|t_i, t_j)$, where i, j are the chosen agents in the auction. Due to the nature of financial markets, agents need to act quickly to new information which is why bidders who receive high signals act quickly to invest in the company given favourable information. This explains the 'hype' on new issue day as we observe a high amount of transactions as investors open and close their positions. However, when the dust settles on the IPO, everyone observes how many of the bidders received high signals which drives how high the first day price rises, then on realisation of how many low signals were received the price falls proportionally. Therefore, suggesting a reason for the phenomena of high returns on the first day as stocks then experience a hangover after on realisation the stock was over-valued.

Part III
Discussion and Conclusion

Throughout this project I have tried to model as accurately as possible an initial public offering auction, comparing my findings to real world events in order to give some context to a heavily theoretical piece of work. I will now culminate a set of extensions which could be made to make the model more accurately replicate real-world behaviour. Of course, an economic model will never perfectly mimic reality due to the imperfect nature of the world.

5 Extensions

5.1 Asymmetric Talent

A strong assumption made in our analysis was to include symmetric values for sensitivity and specificity meaning the bidders are equally good at differentiating between a good and bad firm from their financial health, industry prospects and wider economy trends. Therefore, an interesting extension to the models outlined in this paper would be to introduce the sets $A = \{\alpha_1, ..., \alpha_n\}$ and $B = \{\beta_1, ..., \beta_n\}$, relating to the n bidders in the auction. This would not be difficult to implement if A is observable but when the true value of α_i is only be observable by bidder i, and the rest of the values are known to belong to two distributions, denoted $\tilde{\alpha}$ and $\tilde{\beta}$.

5.2 Repeated Game

Of course, in the economy, the IPO auction is repeating hundreds of times per year as shown in Figure 6, so the model should incorporate this repeated nature. In such a scenario, bidders may choose to stay inactive when they receive low information from the company or when the opposing bidder is sufficiently strong in comparison. Therefore, the model would become a stage game or infinite horizon game where the wealth of the current stage is dependent on previous stages. However, the problem with this extension is that the model would have to accurately define the budget constraint of each bidder which as mentioned in section 2.1, incorporates not just the wealth of the individual but the portfolio the bidder currently holds is a factor which must be considered.

5.3 Final Remarks

The relatively simple models outlined in Parts I and II have provided some interesting conclusions. The complete information model from Part I told us bidders have an incentive to increase the price of the auction when the sum of the competing wealth is less than one third. From this we discovered under certain circumstances, the seller prefers to have one bidder with the majority of the wealth and the rest split amongst the rest, but in these circumstances, the seller may find it better to delay their offering to the stock market in order to maximise their total revenue. An extension from this model was considered in Part

II, which built on the analysis by identifying the importance of good research methods to differentiate between good and bad companies. In addition to this, it also shed some light on the first day return abnormality, from a theoretical point of view, suggesting if inactive bidders in an IPO wait for the first round between the n different bidders to learn more about the expected value of the company before pouncing on the newly publicised stock. Then, the stock can experience a positive shift in the first day of trading,

References

[1] Stitch fix up 16%, author = Alex Wilhelm, howpublished = https://news.crunchbase.com/news/stitch-fix-16-downsized-unicorn-ipo/, note = Accessed: 03/03/19.

[2] World Bank. Gini index (world bank estimate). https://data.worldbank.org/indicator/SI.POV.GINI?end=2016&locations=US&start=1979&view=chart. Accessed: 04/04/19.

[3] Jose M. Bernardo and Adrian F.M. Smith. *Bayesian Theory*. Wiley Series, 1994.

[4] CNBC. Lyft shares fall on second day of trading. https://www.cnbc.com/video/2019/04/01/lyft-shares-fall-on-second-day-of-trading.html. Accessed: 04/04/19.

[5] Ilan Kremer and Kjell G. Nyborg. *Divisible good auctions - the role of allocation rules*. RAND Journal of Economics, 2003.

[6] Vijay Krishna. *Auction Theory*. Academic Print, 2010.

[7] Alexander Ljungqvist. *Chapter III.4: IPO Underpricing*. Handbooks in Finance: Empirical Corporate Finance, 2004.

[8] Tim Loughran and Bill McDonald. *IPO First-Day Returns, Offer Price Revisions, Volatility, and Form S-1 Language*. Journal of Financial Economics, 2013.

[9] NYSE. NYSE total market cap. https://www.nyse.com/market-cap. Accessed: 04/04/19.

[10] Federal Reserve of Economic Data. Real gross domestic product. https://fred.stlouisfed.org/series/GDP/. Accessed: 03/03/19.

[11] Jay R. Ritter and Ivo Welch. *A Review of IPO Activity, Pricing, and Allocations*. Journal of Financial Economics, 2002.

[12] Statista. Largest stock exchanges in europe as of december 2018, by market capitalization. https://www.statista.com/statistics/693587/stock-exchanges-market-capitalization-europe/. Accessed: 03/03/19.

[13] Statista. Number of IPOs in the united states from 1999 to 2018. https://www.statista.com/statistics/270290/number-of-ipos-in-the-us-since-1999/. Accessed: 03/03/19.

[14] Statistics How To. Likelihood ratio. https://www.statisticshowto.datasciencecentral.com/likelihood-ratio/. Accessed: 03/03/19.

Appendices

A Differentiation

In order to find the optimal ϵ for bidder 1 to increase the price of the auction by, where ω is the sum of all completing wealth in the auction, one must differentiate the profit function with respect to ϵ, applying both the product and quotient rule in its derivation. Once the derivative is calculated, one must set it to zero to find the equation for the optimal choice of ϵ. Due to the definition of the auction, one must find the when $\epsilon \geq 0$ as so:

$$
\begin{aligned}
\pi_1 &= (1-\omega-\epsilon)\frac{\omega+\epsilon}{2\omega+\epsilon} \\
\frac{\partial \pi_1(\epsilon)}{\partial \epsilon} &= -1(\frac{\omega+\epsilon}{2\omega+\epsilon}) + (1-\omega-\epsilon)(\frac{\omega}{(2\omega+\epsilon)^2}) \\
&\text{(Setting equal to zero)} \\
\frac{-\omega^2-4\epsilon\omega-\epsilon^2+\omega}{(2\omega+\epsilon)^2} &= 0 \\
-\omega^2-4\epsilon\omega-\epsilon^2+\omega &= 0 \\
&\text{(Using the quadratic formula)} \\
(\epsilon+2\omega+\sqrt{\omega(\omega+1)})(\epsilon+2\omega-\sqrt{\omega(\omega+1)}) &= 0 \\
&\text{(using the Positive Solution)} \\
\epsilon &= -2\omega+\sqrt{\omega(\omega+1)} \\
&\text{(When } \epsilon > 0) \\
-2\omega+\sqrt{\omega(\omega+1)} &> 0 \\
0 \leq \omega &\leq \frac{1}{3}
\end{aligned}
$$

B Probability

- Positive Likelihood Ratio Derivation
  - Pre-test Probability to Pre-test Odds

$$
\begin{aligned}
\textit{Pre-test Odds} &= \frac{\textit{Pre-test Probability}}{1-\textit{Pre-test Probability}} \\
&= \frac{m}{1-m}
\end{aligned}
$$

- Pre-test Odds to Post-test Odds

$$\begin{aligned} \textit{Post-test Odds} &= \textit{Pre-test Odds} * LR_{+} \\ &= \frac{m}{1-m} * \frac{\alpha}{1-\beta} \\ &= \frac{\alpha m}{(1-m)(1-\beta)} \end{aligned}$$

- Post-test Odds to Post-test Probability

$$\begin{aligned} \textit{Post-test Probability} &= \frac{\textit{Post-test Odds}}{\textit{Post-test Odds} + 1} \\ &= \frac{\frac{\alpha m}{(1-m)(1-\beta)}}{\frac{\alpha m}{(1-m)(1-\beta)} + 1} \\ &= \frac{\alpha m}{\alpha m + (1-m)(1-\beta)} \end{aligned}$$

- Negative Likelihood Ratio Derivation
  - Pre-test Probability to Pre-test Odds

$$\begin{aligned} \textit{Pre-test Odds} &= \frac{\textit{Pre-test Probability}}{1 - \textit{Pre-test Probability}} \\ &= \frac{m}{1-m} \end{aligned}$$

  - Pre-test Odds to Post-test Odds

$$\begin{aligned} \textit{Post-test Odds} &= \textit{Pre-test Odds} * LR_{-} \\ &= \frac{m}{1-m} * \frac{1-\alpha}{\beta} \\ &= \frac{(1-\alpha)m}{(1-m)\beta} \end{aligned}$$

  - Post-test Odds to Post-test Probability

$$\begin{aligned} \textit{Post-test Probability} &= \frac{\textit{Post-test Odds}}{\textit{Post-test Odds} + 1} \\ &= \frac{\frac{(1-\alpha)m}{(1-m)\beta}}{\frac{(1-\alpha)m}{(1-m)\beta} + 1} \\ &= \frac{(1-\alpha)m}{(1-\alpha)m + (1-m)\beta} \end{aligned}$$

C Phi and Xi Formula

The formula for ϕ can be found below which indicates the point in which bidder 1 changes from acting aggressively to conservatively.

$$\begin{aligned}\phi = w_1 + \frac{1}{2\mathbb{P}(H|L)}(&\mathbb{P}(H|L)\mathbb{E}(V|H,L) + 3\mathbb{P}(L|L) \\ &- (\sqrt{\mathbb{P}(H|L)^2\mathbb{E}(V|H,L)^2 - 4\mathbb{P}(H|L)^2\mathbb{E}(V|H,L)w_1 + 6w_1\mathbb{P}(H|L)\mathbb{P}(L|L)\mathbb{E}(V|H,L)} \\ &\overline{+4\mathbb{P}(H|L)^2w_1^2 + 12w_1^2\mathbb{P}(L|L)\mathbb{P}(H|L) - 12w_1\mathbb{P}(L|L)\mathbb{P}(H|L)\mathbb{E}(V|L,L) + 9w_1^2\mathbb{P}(L|L)^2})\end{aligned}$$

The formula for ξ, is found by making the two profit functions from the two possible strategies equal to one another. Not much can be taken from this formula other than it is dependent on α, β, m and w_2.

$$\xi = \frac{1}{2(\mathbb{P}(H|H) - 2\mathbb{P}(L|H))}(\mathbb{P}(H|H)\mathbb{E}(V|H,H) - 2\mathbb{P}(L|H)\mathbb{E}(V|H,L) + w_1\mathbb{P}(H|H) + 2w_1\mathbb{P}(L|H) - \sqrt{\zeta})$$

$$\begin{aligned}\zeta = &\mathbb{P}(H|H)^2\mathbb{E}(V|H,H)^2 - 4\mathbb{P}(H|H)\mathbb{P}(L|H)\mathbb{E}(V|H,H)\mathbb{E}(V|H,L) - 2w_1\mathbb{P}(H|H)^2\mathbb{E}(V|H,H)+ \\ &12w_1\mathbb{P}(H|H)\mathbb{P}(L|H)\mathbb{E}(V|H,H) + 4\mathbb{P}(L|H)^2\mathbb{E}(V|H,L)^2 - 12w_1\mathbb{P}(H|H)\mathbb{P}(L|H)\mathbb{E}(V|H,L)+ \\ &8w_1\mathbb{P}(L|H)^2\mathbb{E}(V|H,L) + w_1^2\mathbb{P}(H|H)^2 + 4w_1^2\mathbb{P}(L|H)\mathbb{P}(H|H) + 4w_1^2\mathbb{P}(L|H)^2\end{aligned}$$